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Subsonic Cascade Flutter with Finite Mean Lift

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This paper gives a method of calculating the unsteady aerodynamic forces and predicting the flutter conditions for unstalled two-dimensional cascade blades operating with finite mean lift force in subsonic flows. In this method the unsteady effect of the oscillatory motion of a pressure dipole of a finite steady strength is represented by the equivalent effect of a stationary pressure quadrupole with a fluctuating strength, and all the disturbance quantities are treated on the basis of linearization. Some numerical results are provided for a cascade of flat plates in translational oscillation at finite mean angle of attack. The effect of the Mach number upon the unsteady aerodynamic forces and the marginal flutter conditions are investigated. The critical reduced frequency for a given angle of attack decreases for compressor cascades, while it increases for turbine cascade with increase in the Mach number. The contribution of the finite time mean part of the lift force to the unsteady component of the lift force is heavily influenced by the Mach number. The possibility of negative aerodynamic damping at the resonance conditions is indicated.

Nomenclature

$\bar{C}_L$ = mean lift coefficient defined by Eq. (32)
$\tilde{C}_L, \tilde{C}_{La}, \tilde{C}_{Lr}$ = unsteady lift coefficient defined by Eq. (33)
c = blade chord
$\bar{K}_p, K_{pa}, K_{pr}$ = kernel functions for disturbance pressure [see Eqs. (5–10) and Appendix A]
$\bar{\mathbf{Ku}}, \mathbf{Ku}_a, \mathbf{Ku}_r$ = kernel functions for disturbance velocity [see Eqs. (21–23)]
$\bar{K}_v, K_{va}, K_{vr}$ = y-components of $\bar{\mathbf{Ku}}$, $\mathbf{Ku}_a$, $\mathbf{Ku}_r$ respectively (see Appendix B)
M_o = Mach number of undisturbed flow
p = disturbance pressure
$\bar{p}$ = mean disturbance pressure
s = pitch/chord ratio
t = dimensionless time scaled with respect to c/U_o
U_o = undisturbed velocity of fluid
$\mathbf{u}$ = disturbance velocity
$\bar{\mathbf{u}}$ = mean disturbance velocity
$\tilde{v}, \tilde{v}_a, \tilde{v}_r$ = y-components of $\tilde{\mathbf{u}}$, $\tilde{\mathbf{u}}_a$, $\tilde{\mathbf{u}}_r$ respectively
x, y = dimensionless coordinates scaled with respect to c (see Fig. 2)
$\bar{z}(x)$ = mean position of the blade camber (see Fig. 2)
$\bar{\alpha}_{-\infty}$ = mean angle of attack (see Fig. 2)
γ = angle of stagger (see Fig. 2)
$\delta(\ )$ = Dirac's delta function
$\Delta\bar{p}_B(x)$ = mean pressure jump across a blade
ε = nondimensional small quantity
$\varepsilon a(x)\, e^{i\lambda t}$ = displacement of the blade scaled with respect to c (see Fig. 2)
$\varepsilon\tilde{p}_a e^{i\lambda t}$ = unsteady component of disturbance pressure due to unsteady pressure dipoles
$\varepsilon\tilde{p}_r e^{i\lambda t}$ = unsteady component of disturbance pressure due to unsteady pressure quadrupoles
$\varepsilon\Delta\tilde{p}_B(x)\, e^{i\lambda t}$ = unsteady part of pressure jump across a blade
$\varepsilon\Delta\tilde{p}_{Ba}(x)\, e^{i\lambda t}$ = the first component of $\varepsilon\Delta\tilde{p}_B(x)\, e^{i\lambda t}$
$\varepsilon\Delta\tilde{p}_{Br}(x)\, e^{i\lambda t}$ = the second component of $\varepsilon\Delta\tilde{p}_B(x)\, e^{i\lambda t}$
$\varepsilon\tilde{\mathbf{u}}_a e^{i\lambda t}$ = unsteady component of disturbance velocity due to unsteady pressure dipoles
$\varepsilon\tilde{\mathbf{u}}_r e^{i\lambda t}$ = unsteady component of disturbance velocity due to unsteady pressure quadrupoles
λ = reduced frequency ($= \omega c/U_o$)
σ = interblade phase angle divided by 2π
ρ_o = density of undisturbed fluid
ω = angular frequency of blade oscillation
$\mathscr{R}(\ )$ = real part
$\mathscr{I}(\ )$ = imaginary part

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I. Introduction

IN the past 20 years many studies have been done on the vibration of cascade blades, and the theoretical methods of treating the unstalled cascade flutter for two-dimensional incompressible flows are well developed.[1–3] However, modern axial compressors and turbines are designed to operate in high subsonic or transonic flow conditions, and there exists an urgent need to establish a method of predicting the compressibility effect upon the cascade flutter.

In most of the theoretical work done on the vibration of cascade blades in compressible flows, it has been assumed that the mean deflection of the air through a cascade is very small or zero.[4–6] According to the incompressible flow theories, however, the unstalled bending flutter can not occur unless the mean deflection of the air (in other words, the mean lift force on the blades) is large. In order to account for the bending flutter therefore, it is essential to allow for the finite mean lift force.

Any approach to the problem will encounter a great difficulty in mathematical treatments if it strictly takes account of both

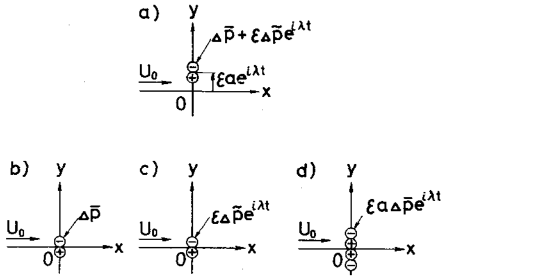

Fig. 1 a) Oscillating dipole, and (b–d) the singularities as its components.

the finite mean lift force and the compressibility effect. This is because the second-order terms of the mean flow disturbance and the cross products of the mean disturbance and unsteady disturbance are in strict sense not negligible compared with the first-order terms of unsteady disturbance, and therefore nonlinear treatments will be generally required. However, some attempts have been made to treat the problem in the scope of linearization. Kaji and Okazaki[7] applied the semiactuator disk theory to the cascade flutter in compressible flows, and provided some useful information on the effect of compressibility. We should bear in mind, however, that the applicability of their theory is limited because of the assumed small spacing of the blades and small interblade phase angle. Recently Nishiyama and Kobayashi[8] have published a theoretical study of flutter of the subsonic cascade based upon the linearized thin airfoil theory. They neglected all the nonlinear terms and the cross products of the unsteady and steady components of the disturbance in solving the differential equations for the steady and unsteady acceleration potentials, but they took account of the effect of relative displacement of the blades with finite mean blade circulation. The linearization in their theory will be justified later in this paper. Unfortunately, however, they mistakenly applied the quasi-steady solution to evaluation of the effect of the relative displacement. In other words the flowfield at a given instant is assumed to be identical with the steady flowfield corresponding to the instantaneous blade positions. We should notice that a quasi-steady solution for the velocity field is valid only for incompressible flow cases, where the disturbance propagates at infinite velocity of sound. When the velocity of sound is finite however, a definite time is required for a disturbance due to movement of a blade to reach a distant field point, and therefore the quasi-steady treatment breaks down.

This paper presents a more complete theory which adopts the linearization like the theory of Nishiyama and Kobayashi,[8] but takes into account the finite velocity of sound in dealing with the effect of the relative movement. The theory is applied to a cascade of flat plates vibrating with translational motion at finite mean angle of attack in subsonic flows, and the effect of compressibility upon the unsteady aerodynamic forces and the marginal flutter conditions are investigated.

II. Basic Concept

The basic assumptions made in the present theory are as follows: 1) the fluid is inviscid and behaves as a perfect gas; 2) the flow is isentropic, subsonic, and two-dimensional; 3) the steady component of the lift force is large compared with the unsteady component; 4) the steady component of the lift force is not too large, so that it can be determined from the linearized equations for the steady disturbance; and 5) the unsteady component of the disturbance is governed by the linearized acoustic wave equation for a uniform flow.

There are some controversial aspects in assumptions 3–5. Assumption 3 is required because the steady part of the lift force does not play an important role upon the unsteady pressure field if it is very small and of the same order as the unsteady part. On the other hand assumptions 4 and 5 are needed in order to simplify the mathematical treatment. To make assumptions 3–5, however, implies that we neglect the nonlinear steady terms and the products of unsteady and steady components of the disturbance with retaining the linear unsteady terms, although the former are not necessarily small in comparison with the latter.

Nevertheless the present model is expected to retain the essential features of the phenomena. What we are most concerned with is the unsteady aerodynamic forces acting on the blades. The contribution of the steady part of the disturbance to the unsteady part of the aerodynamic forces comes mainly from the unsteady upwash induced by the unsteady displacement of the singular points at which the lift force acts. As shown in Appendix C, the proposed perturbation will be rational if the reduced frequency of the blade vibration is small and of the same order as the mean angle of attack.

The concept of pressure dipoles representing lifting surfaces is well established. We can easily find, then, that the effect of small unsteady displacement of a dipole is equivalent to the effect of a quadrupole of unsteady strength. As Fig. 1a shows, consider a pressure dipole in a uniform flow of velocity U_o. The dipole is making a vertical simple harmonic oscillation with an angular frequency ω and a small amplitude εa. Here ε is a nondimensional constant which is small compared with unity, and a denotes a dimensionless length normalized by a characteristic length c, which is later used as the blade chord. Let the strength of the dipole be composed of a finite steady part $\Delta\bar{p}$ and a small unsteady part $\varepsilon\Delta\tilde{p}\, e^{i\lambda t}$, where $\lambda = \omega c/U_o$ is the reduced frequency, and t denotes the dimensionless time scaled with respect to c/U_o. From the assumptions above it follows that the disturbance pressure p should satisfy

$$\diamondsuit^2 p = -(\Delta\bar{p}+\varepsilon\Delta\tilde{p}\, e^{i\lambda t})\delta(x)\delta'(y-\varepsilon a\, e^{i\lambda t}) \tag{1}$$

where $\diamondsuit^2$ is d'Alembertian defined by

$$\diamondsuit^2 = \left(1-M_o^2\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-2M_o^2\frac{\partial^2}{\partial x\,\partial t}-M_o^2\frac{\partial^2}{\partial t^2} \tag{2}$$

and $\delta(\ )$ denotes Dirac's delta function; besides M_o denotes the Mach number of the undisturbed flow, and the coordinates x and y are dimensionless ones normalized by c.

Applying Taylor expansion to the right-hand side of Eq. (1), we get

$$\begin{aligned}-(\Delta\bar{p}+\varepsilon\Delta\tilde{p}\, e^{i\lambda t})\delta(x)\delta'(y-\varepsilon a\, e^{i\lambda t}) &= \\ -\Delta\bar{p}\,\delta(x)\delta'(y)-\varepsilon\Delta\tilde{p}\, e^{i\lambda t}\,\delta(x)\delta'(y)&+ \\ \varepsilon a\Delta\bar{p}\, e^{i\lambda t}\,\delta(x)\delta''(y)+O(\varepsilon^2)\end{aligned} \tag{3}$$

Equation (3) implies that if the terms of order ε^2 are neglected, the effect of the oscillating dipole can be resolved into the following three components, 1) a component due to a fixed dipole of steady strength $\Delta\bar{p}$, 2) a component due to a fixed dipole of fluctuating strength $\varepsilon\Delta\tilde{p}\, e^{i\lambda t}$, and 3) a component due to a fixed quadrupole of fluctuating strength $\varepsilon a\Delta\bar{p}\, e^{i\lambda t}$. Therefore the flowfield of Fig. 1a can be obtained by superposing the three flowfields of Figs. 1b–1d.

III. Disturbance Pressure Field Due to Oscillating Cascade Blades with Finite Mean Lift

As Fig. 2 shows, we consider a cascade of zero-thickness blades with an angle of stagger γ and a pitch/chord ratio s. All the blades are making simple harmonic oscillation with the same amplitude and the same reduced frequency λ but with a constant phase angle $2\pi\sigma$ between one blade to the next. Let the time mean angle of attack $\bar{\alpha}_{-\infty}$ be finite and the amplitude $\varepsilon a(x)$ be small. Then we can consider that the lifting pressure jump across a blade is composed of the finite steady component $\Delta\bar{p}_B(x)$ and the small unsteady component $\varepsilon\Delta\tilde{p}_B(x)\, e^{i\lambda t}$.

Applying the concept in the previous section, each blade may be represented by superposition of the three kinds of singularities, dipoles of steady strength $\Delta\bar{p}_B(x)$, dipoles of fluctuating strength $\varepsilon\Delta\tilde{p}_B(x)\, e^{i\lambda t+2\pi m\sigma i}$, and quadrupoles of fluctuating strength $-\varepsilon a(x)\Delta\bar{p}_B(x)\, e^{i\lambda t+2\pi m\sigma i}$. Here m $(=0, \pm 1, \pm 2, \ldots)$ denotes the blade number. We should note that all the singularities are

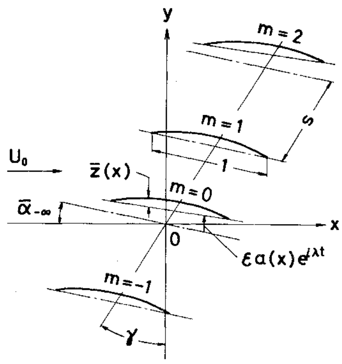

Fig. 2 Coordinate system and notations.

fixed in the stationary parallel line segments denoted by $y = ms\cos\gamma$ and $-\frac{1}{2} \leqq x - ms\sin\gamma \leqq \frac{1}{2}$. Then, ignoring the disturbance effect of order ε^2, we can represent the disturbance pressure $p(x, y, t)$ with the three components as follows

$$p(x,y,t) = \bar{p}(x,y) + \varepsilon\, e^{i\lambda t}\,\tilde{p}_a(x,y) + \varepsilon\, e^{i\lambda t}\,\tilde{p}_r(x,y) \tag{4}$$

where $\bar{p}(x, y)$, $\varepsilon\, e^{i\lambda t}\,\tilde{p}_a(x, y)$, and $\varepsilon\, e^{i\lambda t}\,\tilde{p}_r(x, y)$ are the disturbance components due to the steady dipoles, unsteady dipoles, and unsteady quadrupoles, respectively. According to the basic assumptions each component should independently satisfy the linearized acoustic wave equation, and we can ultimately obtain the integral representations for each component as follows

$$\bar{p}(x,y) = \int_{-1/2}^{1/2} \Delta\bar{p}_B(\xi)\bar{K}_p(x-\xi, y)\, d\xi \tag{5}$$

$$\tilde{p}_a(x,y) = \int_{-1/2}^{1/2} \Delta\tilde{p}_B(\xi) K_{pa}(x-\xi, y)\, d\xi \tag{6}$$

$$\tilde{p}_r(x,y) = \int_{-1/2}^{1/2} \Delta\bar{p}_B(\xi) a(\xi) K_{pr}(x-\xi, y)\, d\xi \tag{7}$$

where the kernel functions $\bar{K}_p(x, y)$, $K_{pa}(x, y)$, and $K_{pr}(x, y)$ are the solutions of the following differential equations

$$\nabla^{*2}\bar{K}_p(x,y) = -\sum_{m=-\infty}^{\infty} \delta(x - ms\sin\gamma)\delta'(y - ms\cos\gamma) \tag{8}$$

$$\Diamond^{*2}K_{pa}(x,y) = -\sum_{m=-\infty}^{\infty} \delta(x - ms\sin\gamma)\delta'(y - ms\cos\gamma)\, e^{2\pi m\sigma i} \tag{9}$$

$$\Diamond^{*2}K_{pr}(x,y) = \sum_{m=-\infty}^{\infty} \delta(x - ms\sin\gamma)\delta''(y - ms\cos\gamma)\, e^{2\pi m\sigma i} \tag{10}$$

Here ∇^{*2} and $\Diamond^{*2}$ are the differential operators defined by

$$\nabla^{*2} = (1 - M_o^{\,2})\,\partial^2/\partial x^2 + \partial^2/\partial y^2 \tag{11}$$

and

$$\Diamond^{*2} = \nabla^{*2} - 2i\lambda M_o^{\,2}\,\partial/\partial x + \lambda M_o^{\,2} \tag{12}$$

respectively. A brief description of the solution of Eqs. (8–10) as well as the explicit expressions of the kernel functions are given in Appendix A.

IV. Disturbance Velocity

According to basic assumptions 1–5, the disturbance velocity $\mathbf{u}(x, y, t)$ should satisfy the linearized Euler equation of motion

$$D\mathbf{u}/Dt = -\nabla p/(\rho_o U_o) \tag{13}$$

where $D/Dt = \partial/\partial t + \partial/\partial x$, and ρ_o denotes the density of undisturbed fluid. Then, since the pressure p is resolved into the three components as shown by Eq. (4), the disturbance velocity $\mathbf{u}(x, y, t)$, likewise, can be resolved into the three components as follows

$$\mathbf{u}(x,y,t) = \bar{\mathbf{u}}(x,y) + \varepsilon\, e^{i\lambda t}\,\tilde{\mathbf{u}}_a(x,y) + \varepsilon\, e^{i\lambda t}\,\tilde{\mathbf{u}}_r(x,y) \tag{14}$$

where $\bar{\mathbf{u}}(x, y)$, $\tilde{\mathbf{u}}_a(x, y)$, and $\tilde{\mathbf{u}}_r(x, y)$ should separately satisfy Euler equation of motion as follows

$$\partial\bar{\mathbf{u}}/\partial x = -\nabla\bar{p}/(\rho_o U_o) \tag{15}$$

$$(i\lambda + \partial/\partial x)\tilde{\mathbf{u}}_a = -\nabla\tilde{p}_a/(\rho_o U_o) \tag{16}$$

$$(i\lambda + \partial/\partial x)\tilde{\mathbf{u}}_r = -\nabla\tilde{p}_r/(\rho_o U_o) \tag{17}$$

Using Eqs. (5–7) and integrating Eqs. (15–17), we can finally reach the following integral forms

$$\bar{\mathbf{u}}(x,y) = \frac{1}{\rho_o U_o}\int_{-1/2}^{1/2} \Delta\bar{p}_B(\xi)\overline{\mathbf{Ku}}(x-\xi, y)\, d\xi \tag{18}$$

$$\tilde{\mathbf{u}}_a(x,y) = \frac{1}{\rho_o U_o}\int_{-1/2}^{1/2} \Delta\tilde{p}_B(\xi)\,\mathbf{Ku}_a(x-\xi, y)\, d\xi \tag{19}$$

$$\tilde{\mathbf{u}}_r(x,y) = \frac{1}{\rho_o U_o}\int_{-1/2}^{1/2} a(\xi)\Delta\bar{p}_B(\xi)\mathbf{Ku}_r(x-\xi, y)\, d\xi \tag{20}$$

where the kernel functions $\overline{\mathbf{Ku}}(x, y)$, $\mathbf{Ku}_a(x, y)$, and $\mathbf{Ku}_r(x, y)$ are defined by

$$\overline{\mathbf{Ku}}(x,y) = -\int_{-\infty}^{x} \nabla\bar{K}_p(\zeta, y)\, d\zeta \tag{21}$$

$$\mathbf{Ku}_a(x,y) = -e^{-i\lambda x}\int_{-\infty}^{x} e^{i\lambda\zeta}\,\nabla K_{pa}(\zeta, y)\, d\zeta \tag{22}$$

$$\mathbf{Ku}_r(x,y) = -e^{-i\lambda x}\int_{-\infty}^{x} e^{i\lambda\zeta}\,\nabla K_{pr}(\zeta, y)\, d\zeta \tag{23}$$

The expressions of the y-components of the kernel functions in the forms convenient for calculation are given in Appendix B.

V. Determination of Lifting Force

Let the time mean position of the zeroth blade camberline be denoted by $y = \bar{z}(x)$. Then its instantaneous position is given by

$$y = \bar{z}(x) + \varepsilon a(x)\, e^{i\lambda t} \tag{24}$$

Since the singularities for the zeroth blade are placed in the line $y = 0$ instead of $y = \bar{z}(x)$, we can evaluate the upwash velocity at the blade by

$$v(x, \varepsilon a(x)\, e^{i\lambda t}, t) =$$
$$\bar{v}(x,0) + \varepsilon\, e^{i\lambda t}\{a(x)\,\partial\bar{v}(x,0)/\partial y + \tilde{v}_a(x,0) + \tilde{v}_r(x,0)\} + O(\varepsilon^2) \tag{25}$$

where v, $\bar{v}$, $\tilde{v}_a$, and $\tilde{v}_r$ are the y-components of $\mathbf{u}$, $\bar{\mathbf{u}}$, $\tilde{\mathbf{u}}_a$, and $\tilde{\mathbf{u}}_r$ respectively. Then the condition of tangency of the streamline along the blade camberline can be expressed by

$$\begin{gathered}\bar{v}(x,0)/U_o = \bar{z}'(x)\\ \{a(x)\,\partial\bar{v}(x,0)/\partial y + \tilde{v}_a(x,0) + \tilde{v}_r(x,0)\}/U_o = i\lambda a(x) + a'(x)\end{gathered} \tag{26}$$

Combining Eq. (26) with Eqs. (18–20), we get the following integral equations for determining $\Delta\bar{p}_B(x)$ and $\Delta\tilde{p}_B(x)$

$$\frac{1}{\rho_o U_o^{\,2}}\int_{-1/2}^{1/2} \Delta\bar{p}_B(\xi)\bar{K}_v(x-\xi, 0)\, d\xi = \bar{z}'(x) \tag{27}$$

$$\frac{1}{\rho_o U_o^{\,2}}\int_{-1/2}^{1/2} \Delta\tilde{p}_{Ba}(\xi) K_{va}(x-\xi, 0)\, d\xi = i\lambda a(x) + a'(x) \tag{28}$$

$$\int_{-1/2}^{1/2} \Delta\tilde{p}_{Br}(\xi) K_{va}(x-\xi, 0)\, d\xi + \int_{-1/2}^{1/2} \Delta\bar{p}_B(\xi) a(\xi) K_{vr}(x-\xi, 0)\, d\xi + a(x)\int_{-1/2}^{1/2} \Delta\bar{p}_B(\xi)\frac{\partial}{\partial y}\bar{K}_v(x-\xi, 0)\, d\xi = 0 \tag{29}$$

where $\bar{K}_v$, K_{va}, and K_{vr} are the y-components of $\overline{\mathbf{Ku}}$, $\mathbf{Ku}_a$, and $\mathbf{Ku}_r$, respectively. Furthermore the unsteady part of the pressure jump across the blade is for convenience resolved into two components as follows

$$\Delta\tilde{p}_B(x) = \Delta\tilde{p}_{Ba}(x) + \Delta\tilde{p}_{Br}(x) \tag{30}$$

Equation (28) implies that the first component $\varepsilon\Delta\tilde{p}_{Ba}(x)\, e^{i\lambda t}$ corresponds to the unsteady pressure jump in the case of the blade oscillation at zero mean lift force. On the other hand, Eq. (29) means that the second component $\varepsilon\Delta\tilde{p}_{Br}(x)\, e^{i\lambda t}$ is the unsteady pressure jump due to the unsteady upwash caused by the unsteady quadrupoles or, in other words, caused by relative displacement of the dipoles of finite steady strength loaded on the blades.

The amplitude function $a(x)$ depends upon the oscillation mode. If the blade oscillation is of translational motion, we can put $a(x) = 1$. On the other hand, if the oscillation is of pitching motion about the midchord, then $a(x) = x$. For the oscillation of the coupled mode of translation and pitching about an arbitrary axis position, the aerodynamic force and moment can be obtained by linear combination of those for the two fundamental cases mentioned above. Therefore we have only to solve the integral equations [Eqs. (28) and (29)] for $a(x) = 1$ and $a(x) = x$.

The most preferable method to solve the integral equations [Eqs. (27–29)] will be to expand $\Delta\bar{p}_B(x)$, $\Delta\tilde{p}_{Ba}(x)$, and $\Delta\tilde{p}_{Br}(x)$ in terms of the so-called Glauert series which satisfies the Kutta condition at the trailing edge and possesses the proper type of infinity at the leading edge. Since this method has been widely used by others,[2,4,5,8,9] no detailed description will be necessary here. Only the first four terms in each series have been considered in obtaining the numerical results in the present paper.

VI. Resonance Condition

As is well known,[4–9] the compressibility of the fluid gives rise to the so-called resonance phenomena in unsteady flows through cascades. With the notations in the present paper, the resonance condition can be expressed as

$$\Omega_v^2 = \frac{4\pi^2\cos^2\gamma}{\beta_a^2 s^2}\left[(v+\sigma)^2 - \frac{M_a^2}{\beta_a^2}\left\{(v+\sigma)\tan\gamma + \frac{s\lambda}{2\pi\cos\gamma}\right\}^2\right] = 0 \qquad (31)$$

As for the definitions of the notations, see Appendix A. It can be seen from Appendices A and B that the kernel functions $K_{pa}(x, y)$, $K_{pr}(x, y)$, $\mathbf{Ku}_a(x, y)$, and $\mathbf{Ku}_r(x, y)$ become infinite at the resonance conditions since they involve the terms of order Ω_v^{-1}. It follows from Eq. (28) that the first component of the unsteady lift force $\Delta\tilde{p}_{Ba}(x)$ vanishes at the resonance condition. This confirms the results in Refs. 4 and 5. It should be noticed, however, that the second component $\Delta\tilde{p}_{Br}(x)$ does not necessarily vanish, but in general remains finite at the resonance condition. This is because not only $K_{va}(x-\xi, 0)$ in the first term of Eq. (29) but also the second term involving $K_{vr}(x-\xi, 0)$ tend to infinity of order Ω_v^{-1}. In this condition the infinite induced velocity due to the quadrupoles represented by the second term of Eq. (29) can only be cancelled by the opposite infinite induced velocity due to the second component of the unsteady lift force $\Delta\tilde{p}_{Br}(x)$ of nonzero value represented by the first term of Eq. (29). As seen later from the numerical results, this brings about the possibility of the resonance flutter which can occur even at very small but nonzero mean angles of attack.

VII. Cascade of Flat Plates of Translational Oscillation

In this paper we shall not attempt an extensive study, but shall concern ourselves only with a flat plate cascade of translational oscillation. This is because this case, in which the finite mean lift force plays an important role on occurrence of flutter, is the most fundamental case to be explored.

Let us denote the time mean angle of attack by $\bar{\alpha}_{-\infty}$, which is the angle of the time mean direction of the chord-line with respect to the far upstream flow direction. Then $\bar{z}'(x) = -\bar{\alpha}_{-\infty}$ in Eq. (27). Since we are dealing with translational oscillation, we can put $a(x) = 1$, and therefore ε denotes the amplitude of vertical displacement divided by the chord length. We define the mean lift coefficient $\bar{C}_L$ by

$$\bar{C}_L = \frac{1}{\frac{1}{2}\rho_o U_o^2}\int_{-1/2}^{1/2} \Delta\bar{p}_B(x)\,dx \qquad (32)$$

and the unsteady lift coefficients $\tilde{C}_L$, $\tilde{C}_{La}$, and $\tilde{C}_{Lr}$ by

$$\begin{bmatrix}\tilde{C}_L\\ \tilde{C}_{La}\\ \tilde{C}_{Lr}\end{bmatrix} = \int_{-1/2}^{1/2}\begin{bmatrix}\varepsilon\Delta\tilde{p}_B(x)\\ \varepsilon\Delta\tilde{p}_{Ba}(x)\\ \varepsilon\Delta\tilde{p}_{Br}(x)\end{bmatrix} dx/(\pi\rho_o U_o q) \qquad (33)$$

where $q\,e^{i\lambda t} = i\varepsilon U_o \lambda\, e^{i\lambda t}$ denotes the velocity of the blade oscillation. We should note that $\bar{C}_L/\bar{\alpha}_{-\infty}$ and $\tilde{C}_{Lr}/\bar{\alpha}_{-\infty}$ are independent of $\bar{\alpha}_{-\infty}$ because of linearization. From Eq. (30) it follows that

$$\tilde{C}_L = \tilde{C}_{La} + \bar{\alpha}_{-\infty}(\tilde{C}_{Lr}/\bar{\alpha}_{-\infty}) \qquad (34)$$

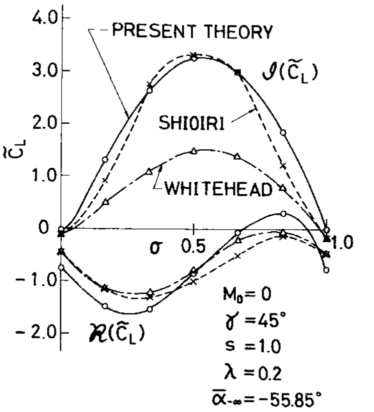

Fig. 3 Comparison with the previous incompressible flow theories by Shioiri[1] and Whitehead.[2]

As is well known,[2] if the mechanical damping is neglected, the condition for the vibration to be self-excited is given by $\mathscr{R}(\tilde{C}_L) > 0$, where $\mathscr{R}(\;)$ denotes the real part. Therefore the marginal flutter condition is represented by

$$\mathscr{R}(\tilde{C}_L) = \mathscr{R}(\tilde{C}_{La}) + \bar{\alpha}_{-\infty}\mathscr{R}(\tilde{C}_{Lr}/\bar{\alpha}_{-\infty}) = 0 \qquad (35)$$

and hence the critical angle of attack for flutter is given by

$$\bar{\alpha}_{-\infty cr} = -\mathscr{R}(\tilde{C}_{La})/\mathscr{R}(\tilde{C}_{Lr}/\bar{\alpha}_{-\infty}) \qquad (36)$$

In the following, using some numerical results, we shall investigate the characteristics of the unsteady lift force coefficients $\tilde{C}_{La}$ and $\tilde{C}_{Lr}/\bar{\alpha}_{-\infty}$, and the dependence of the flutter conditions upon various parameters. The calculations have been conducted on the electronic computer FACOM 230-60 in Kyushu University.

Comparison with the Incompressible Exact Theories

The present theory linearizes not only the unsteady component of the disturbance properties but also the steady component, and furthermore it neglects the cross products of the steady and unsteady components. In order to check the validity of this treatment, we shall compare the results for the unsteady lift force in an incompressible flow obtained by the present theory with those by the previous incompressible flow theories.[1,2] Figure 3 shows the dependence of $\tilde{C}_L$ upon the interblade phase difference σ in the case of an incompressible flow. In spite of the rather large mean angle of attack of $\bar{\alpha}_{-\infty} = 55.85°$ the agreement is satisfactory especially for the real part which determines the flutter condition. We should notice that the Shioiri's theory[1] omits the terms of $\bar{\alpha}_{-\infty}^2$ in calculating the blade force. This may be one of the reasons why the present theory shows better agreement with the theory by Shioiri[1] than with that by Whitehead.[2]

First Component of the Unsteady Lift Coefficient $\tilde{C}_{La}$

Since this component corresponds to the unsteady lift coefficient for the zero mean lift conditions, it has been already investigated by others.[4,5] However, in order to facilitate the comparison with the second component, we had better reconfirm its characteristics. Figure 4 shows how the relation between $\tilde{C}_{La}$ and the interblade phase angle $2\pi\sigma$ is influenced by the Mach number M_o. In the following figures the resonance points are marked by R.P. In all cases shown in this paper where the reduced frequency λ is not large, the resonance occurs for a given nonzero Mach number at a pair of σ. In other words in the present cases, Eq. (31) can be satisfied only when $v = 0$, giving two real roots of σ. The superresonance region where the nondecaying disturbance occurs corresponds to the region of σ between the resonance points in which the in-phase condition; i.e., $\sigma = 0$ or $\sigma = 1$, is included, while the other region bounded by the resonance points where the in-phase condition does not exist is the subresonance region, where all the unsteady pressure disturbance decay at far field.

First, we can see that $|\tilde{C}_{La}|$ becomes large as M_o increases, except near the resonance points. Second, the absolute value of

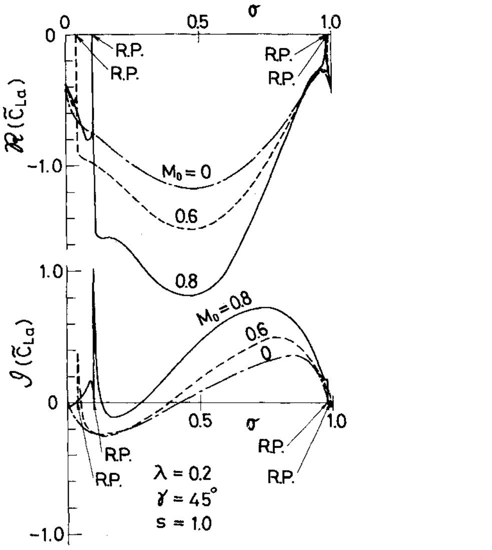

Fig. 4 Dependence of the first component of the unsteady lift coefficient $\tilde{C}_{La}$ upon the phase difference σ and the Mach number M_o.

the real part of $\tilde{C}_{La}$ takes a maximum value near the out of phase point $\sigma = \frac{1}{2}$. Third, $\tilde{C}_{La}$ falls to zero almost abruptly at the resonance points. Finally, the real part of $\tilde{C}_{La}$ never becomes positive at any value of σ. This implies that the pure bending flutter does not occur at zero mean deflection.

Second Component of the Unsteady Lift Coefficient $\tilde{C}_{Lr}$

Since $\tilde{C}_{Lr}$ is evidently proportional to the mean lift coefficient $\bar{C}_L$, it would be better to examine the normalized value $\tilde{C}_{Lr}/\bar{C}_L$. An example in Fig. 5 shows the variation of $\tilde{C}_{Lr}/\bar{C}_L$ with σ and its dependence upon M_o. In subresonance region the real part $\tilde{C}_{Lr}/\bar{C}_L$ takes a positive maximum value near $\sigma = \frac{1}{4}$ and a negative minimum value near $\sigma = \frac{3}{4}$. The positive maximum value for positive stagger angle γ decreases as M_o increases, while the negative minimum value for positive stagger angle γ slightly decreases with increase in M_o. The zero point of $\mathcal{R}(\tilde{C}_{Lr}/\bar{C}_L)$ at which it changes the sign usually lies between $\frac{1}{4} < \sigma < \frac{1}{2}$ for positive stagger angle γ, and it moves towards smaller σ as M_o increases.

As for the effect of the reduced frequency λ upon $\tilde{C}_{Lr}/\bar{C}_L$, though not illustrated by figures in this paper, it is found that $|\tilde{C}_{Lr}/\bar{C}_L|$ is heavily influenced by λ, and it generally decreases as λ increases as far as the subresonance region is concerned.

Very sharp fall in $\tilde{C}_{Lr}/\bar{C}_L$ occurs at the resonance points, but as previously suggested, $\mathcal{R}(\tilde{C}_{Lr}/\bar{C}_L)$ is found to remain finite there. It may also be worthwhile to note that in the super-

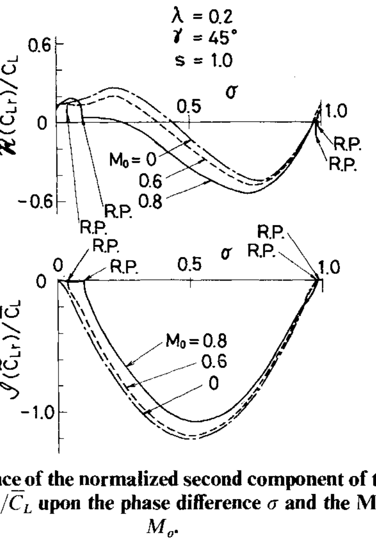

Fig. 5 Dependence of the normalized second component of the unsteady lift coefficient $\tilde{C}_{Lr}/\bar{C}_L$ upon the phase difference σ and the Mach number M_o.

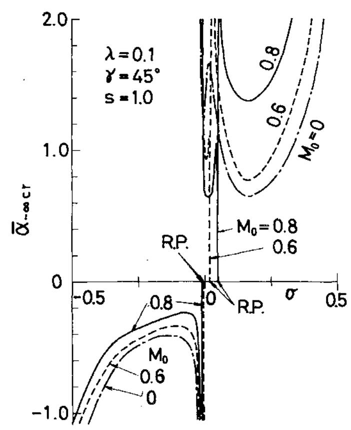

Fig. 6 Variation of the critical mean angle of attack $\bar{\alpha}_{-\infty cr}$ with the phase difference σ and the Mach number M_o.

resonance region the imaginary part of $\tilde{C}_{Lr}/\bar{C}_L$ is very small. It means that the second component of the unsteady pressure jump $\varepsilon\Delta\tilde{p}_{Br}e^{i\lambda t}$ is nearly in phase with the blade velocity in the superresonance region. When γ is positive, $\mathcal{R}(\tilde{C}_{Lr}/\bar{C}_L)$ has a positive maximum in the superresonance region too. This maximum value is found to overtake that in the subresonance region when either M_o or λ becomes large.

Critical Angle of Attack $\bar{\alpha}_{-\infty cr}$

Figure 6 shows an example of the variation of the critical angle of attack $\bar{\alpha}_{-\infty cr}$ with σ and M_o. In the subresonance region, $\bar{\alpha}_{-\infty cr}$ for a given M_o takes a positive minimum value around $\sigma = 0.15$ and a negative maximum value around $\sigma = -0.15$. Since in the present case of a cascade of flat plates the negative angle of attack with a positive angle of stagger γ corresponds to a turbine cascade, we can say that the critical angle of attack becomes minimum at a phase lag angle of about 60° for both the compressor and turbine cascades, as far as they are in subresonance condition. It should also be pointed out, however, that the phase lag σ at which $\bar{\alpha}_{-\infty cr}$ becomes minimum, decreases with increase in M_o in the case of turbine cascades. It should be noted also that the minimum of $\bar{\alpha}_{-\infty cr}$ in the subresonance region for the turbine cascade is smaller than that for the compressor cascade, and that the former decreases with increase in M_o, while the latter increases with M_o.

In the superresonance region, which is confined to a very narrow region around $\sigma = 0$ in the present example of not large λ, the critical angle of attack $\bar{\alpha}_{-\infty cr}$ also takes a positive minimum value at a positive value of σ for $\gamma > 0$. This minimum value for a given M_o is larger than that in the subresonance region, as long as either M_o or λ is small. The minimum value of $\bar{\alpha}_{-\infty cr}$ in the superresonance region is, however, almost insensitive to λ, though not illustrated by figures in this paper. On the other hand, as is seen in Fig. 6, it decreases with increase in M_o. Consequently, the minimum value of $\bar{\alpha}_{-\infty cr}$ in the superresonance region is surpassed by that in the subresonance region when either M_o or λ becomes large. It should be noted, however, that even in the case of $M_o = 0.8$, the minimum value of $\bar{\alpha}_{-\infty cr}$ in the superresonance region is so large that the superresonance flutter seems unlikely to occur for a conventional compressor cascade operating at not too large angle of attack.

At the resonance points the critical angle of attack, $\bar{\alpha}_{-\infty cr}$ sharply falls to zero. This means that at the resonance points there is a possibility of occurrence of flutter even at a very small mean angle of attack. However it should also be noticed that even a small departure from the resonance points can give rise to recovery from the flutter of this type.

Critical Reduced Frequency λ_{cr}

For a given $\bar{\alpha}_{-\infty}$ there is a value of λ for which the minimum of $\bar{\alpha}_{-\infty cr}$ coincides with the given $\bar{\alpha}_{-\infty}$. This is the critical reduced frequency λ_{cr}, and the flutter occurs at λ lower than λ_{cr}

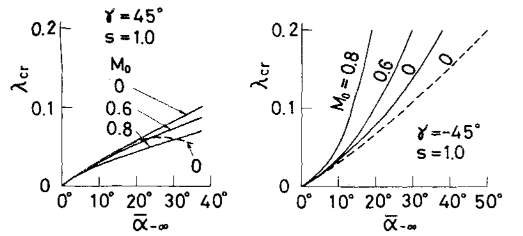

Fig. 7 Variation of the critical reduced frequency λ_{cr} with the mean angle of attack $\bar{\alpha}_{-\infty}$ and the Mach number M_o; ---- incompressible flow theory by Whitehead.[2]

of a given $\bar{\alpha}_{-\infty}$. Figure 7 shows the variation of λ_{cr} with $\bar{\alpha}_{-\infty}$ and M_o for $\gamma = \pm 45°$ and $s = 1.0$. Hereafter, the resonance flutter is not considered.

First, we see that λ_{cr} for the compressor cascade ($\gamma > 0$) is generally smaller than that for the turbine cascade. Second, λ_{cr} decreases as M_o increases for the compressor cascade, while λ_{cr} increases with M_o for the turbine cascade. In Fig. 7 the results by Whitehead[2] for the incompressible flow case are also shown with broken lines. Agreement with the present results for $M_o = 0$ is satisfactory unless $\bar{\alpha}_{-\infty}$ is too large.

It should be noted that the mean lift coefficient $\bar{C}_L$ increases with M_o. Therefore in order to evaluate the effect of compressibility upon λ_{cr} for a given mean blade loading, λ_{cr} should be plotted against $\bar{C}_L/s$. Figure 8 shows how the dependence of λ_{cr} upon $\bar{C}_L/s$ is influenced by the Mach number. From this standpoint then, we can say that the compressibility reduces λ_{cr} not only for the compressor cascade, but also for the turbine cascade unless the mean blade loading is too large. In this figure the results of the semi-actuator disk theory by Kaji and Okazaki[7] are also shown with broken lines. Fairly good agreement can be seen in the present case of the pitch/chord ratio of $s = 1$, as far as $\bar{\alpha}_{-\infty}$ is not too large.

Figure 9 gives the results for a larger pitch/chord ratio of $s = 1.5$. It is seen, however, that the accuracy of the semi-actuator disk theory deteriorates for large pitch/chord ratio. Comparison between Figs. 8 and 9 indicates that λ_{cr} decreases with increase in s. It should also be noted that in Fig. 9 the effect of M_o upon λ_{cr} is reversed when $\bar{C}_L/s$ becomes large.

VIII. Conclusion

The effect of a pressure dipole of a finite steady strength making small oscillatory motion is shown to be equivalent to the combined effect of a stationary dipole of the steady strength and a stationary quadrupole of a fluctuating strength. This concept can be successfully applied to evaluating the effect of relative motion of the blades with finite mean lift in compressible flows. The present method of analysis is found very useful for predicting the compressibility effect upon the unsteady aerodynamic forces and the marginal flutter conditions for a cascade of unstalled blades operating at finite but not too large mean angle of attack.

The proposed method has been applied to a cascade of flat plates in translational oscillation, and the following conclusions have been obtained. The critical reduced frequency for a given mean blade loading decreases with increase in the Mach number

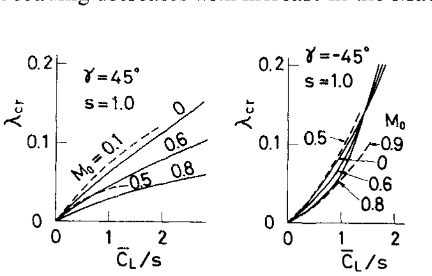

Fig. 8 Variation of the critical reduced frequency λ_{cr} with the mean lift coefficient per pitch $\bar{C}_L/s$ and the Mach number M_o; ---- semi-actuator disk theory by Kaji and Okazaki.[7]

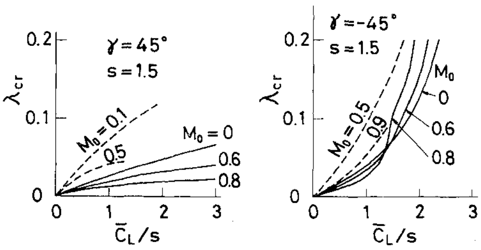

Fig. 9 Variation of the critical reduced frequency λ_{cr} with the mean lift coefficient per pitch $\bar{C}_L/s$ and the Mach number M_o; ---- semi-actuator disk theory by Kaji and Okazaki.[7]

for turbine cascades with small blade loading as well as for compressor cascades. For turbine cascades, however, this effect of the Mach number decreases as the blade loading increases. The critical reduced frequency for a given mean angle of attack increases with the Mach number for turbine cascades, while it decreases with increase in the Mach number for compressor cascades. In the ordinary range of the angle of attack the super-resonance flutter is unlikely to occur. At the resonance conditions the aerodynamic damping can become negative even for a very small but nonzero mean angle of attack.

Appendix A Kernel Functions $\bar{K}_p(x, y)$, $K_{pa}(x, y)$ and $K_{pr}(x, y)$

The solution to Eq. (8) gives the disturbance pressure field due to a row of pressure dipoles of unit strength, which is well known in the classical two-dimensional subsonic cascade theory. We can express it as

$$\bar{K}_p(x, y) = -(\beta \cos\gamma \cdot \sin 2A - \sin\gamma \cdot \sin 2B)/\{2s\beta_a^2(\cosh 2B - \cos 2A)\} \quad \text{(A1)}$$

where

$$\beta_a^2 = 1 - M_a^2, \quad M_a = M_o \cos\gamma, \quad \beta^2 = 1 - M_o^2$$

$$A = \pi(x \sin\gamma + y\beta^2 \cos\gamma)/(s\beta_a^2),$$

$$B = \pi\beta(x \cos\gamma - y \sin\gamma)/(s\beta_a^2)$$

The solution to Eq. (9) corresponds to the disturbance pressure field due to a row of pressure dipoles of fluctuating strength with unit amplitude and phase difference of $2\pi\sigma$. Not a few papers[4-6,8,9] have given the solution in various forms. The form most convenient for calculation will be that in terms of rapidly converging Fourier series which Kaji and Okazaki[9] have derived starting from superposition of the single dipole solutions of Hankel function form. Here is shown a more efficient method of obtaining the solution in the Fourier series form. Assume $K_{pa}(x, y)$ in the Fourier integral form of

$$K_{pa}(x, y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(\alpha, \mu)\, e^{i\alpha x + i\mu y}\, d\alpha\, d\mu \quad \text{(A2)}$$

Then it follows from Eq. (9) that

$$H(\alpha, \mu) = \left[\frac{i\mu}{\beta^2} \Big/ \left\{\left(\alpha - \frac{M_o^2\lambda}{\beta^2}\right)^2 + \left(\mu^2 - \frac{M_o^2\lambda^2}{\beta^2}\right)\frac{1}{\beta^2}\right\}\right] \times \sum_{m=-\infty}^{\infty} \exp\{-ims(\alpha \sin\gamma + \mu \cos\gamma - 2\pi\sigma/s)\} \quad \text{(A3)}$$

Then utilizing the identity[10] of

$$\sum_{m=-\infty}^{\infty} \exp\{-im\pi(\mu + \alpha \tan\gamma - 2\pi\sigma/s \cos\gamma)/(\pi/s \cos\gamma)\} = (2\pi/s \cos\gamma) \sum_{\nu=-\infty}^{\infty} \delta(\mu + \alpha \tan\gamma - 2\pi[\nu + \sigma]/s \cos\gamma)$$

and conducting integration with respect to α and μ, we obtain

$$K_{pa}(x, y) = \frac{i}{2s\beta_a^2} e^{i\phi\eta} \sum_{\nu=-\infty}^{\infty} \left(\frac{q_\nu \beta^2 \cos^2\gamma - M_a^2\lambda \sin\gamma}{\beta_a^2\Omega_\nu} - i\,\mathrm{sgn}\,\eta \cdot \sin\gamma\right) e^{2i(\nu+\sigma)A - |\eta|\Omega_\nu} \quad \text{(A4)}$$

where

$$\phi = M_a^2\lambda/\beta_a^2, \quad \eta = x - y\tan\gamma, \quad q_\nu = 2\pi(\nu+\sigma)/s$$

and

$$\Omega_\nu^2 = \{q_\nu^2\cos^2\gamma - (q_\nu \sin\gamma + \lambda)^2 M_a^2/\beta_a^2\}/\beta_a^2 \qquad \text{(A5)}$$

Here Ω_ν is a positive real number or an imaginary number with the same sign as $i(q_\nu \sin\gamma + \lambda)$, according as $\Omega_\nu^2 > 0$ or $\Omega_\nu^2 < 0$. We can further modify the expression (A4) into a form more convenient for numerical computation of the disturbance pressure as will be worked out for $K_{va}(x, y)$ in Appendix B. Because of the limited space, however, it is not shown here.

The solution to Eq. (10) gives the disturbance pressure field due to a row of pressure quadrupoles of fluctuating strength with unit amplitude and phase difference of $2\pi\sigma$. It is evident from Eqs. (9) and (10) that

$$K_{pr}(x, y) = -\partial K_{pa}(x, y)/\partial y \qquad \text{(A6)}$$

The expression of $K_{pr}(x, y)$ in a Fourier series form similar to Eq. (A4) is omitted here.

Appendix B Kernel Functions $\bar{K}_v(x, y)$, $K_{va}(x, y)$, and $K_{vr}(x, y)$

Using Eqs. (A1), (A4), and (A6), we can conduct integration in Eqs. (21–23). Only the results for the y-components are given in the following:

$$\bar{K}_v(x, y) = -\frac{\beta}{2s\beta_a^2}\left(\beta\cos\gamma + \frac{\sin\gamma\cdot\sin 2A + \beta\cos\gamma\cdot\sinh 2B}{\cosh 2B - \cos 2A}\right) \qquad \text{(B1)}$$

$$K_{va}(x, y) = K_{va}^{(0)}(x, y) + K_{va}^{(S)}(x, y) + K_{va}^{(R)}(x, y) \qquad \text{(B2)}$$

$$K_{vr}(x, y) = K_{vr}^{(0)}(x, y) + K_{vr}^{(S)}(x, y) + K_{vr}^{(R)}(x, y) \qquad \text{(B3)}$$

Here $K_{va}^{(0)}$ and $K_{vr}^{(0)}$ denote the Fourier components of the longest wavelength in the y direction. The second parts $K_{va}^{(S)}$ and $K_{vr}^{(S)}$ are the singular parts expressed in the closed forms. The third parts $K_{va}^{(R)}$ and $K_{vr}^{(R)}$ denote the regular parts expressed as uniformly converging Fourier series. Details of the expressions are as follows:

$$K_{va}^{(0)}(x, y) = -\frac{1}{2s\beta_a^2}(1+\operatorname{sgn}\eta) I_o e^{-i\lambda\eta + iq_o y/\cos\gamma} + \frac{1}{2s\beta_a^2}(I_o \operatorname{sgn}\eta - iJ_o) e^{i\phi\eta + 2i\sigma A - |\eta|\Omega_o} \qquad \text{(B4)}$$

$$K_{va}^{(S)}(x, y) = -\frac{1}{2s\beta_a^2}(1+\operatorname{sgn}\eta)\beta^2\cos\gamma\cdot e^{-i\lambda\eta + iq_o y/\cos\gamma} + \frac{1}{4s\beta_a^2} e^{i\phi\eta}\Bigg[e^{2i\sigma A - 2|B|\psi}\Big\{(-\beta^2\cos\gamma\cdot\operatorname{sgn}\eta + i\beta\sin\gamma)\times R\coth(C) - i\frac{s\lambda\beta_a^2}{\pi\beta} S\coth(C)\Big\} + e^{2i\sigma A + 2|B|\psi}\Big\{(-\beta^2\cos\gamma\cdot\operatorname{sgn}\eta - i\beta\sin\gamma) R\coth(D) - i\frac{s\lambda\beta_a^2}{\pi\beta} S\coth(D)\Big\}\Bigg] \qquad \text{(B5)}$$

$$K_{va}^{(R)}(x, y) = -\frac{1}{2s\beta_a^2}(1+\operatorname{sgn}\eta) e^{-i\lambda\eta}\sum_{\substack{\nu=-\infty\\ \neq 0}}^{\infty}(I_\nu + \beta^2\cos\gamma)\times e^{iq_\nu y/\cos\gamma} + \frac{1}{2s\beta_a^2} e^{i\phi\eta}\sum_{\substack{\nu=-\infty\\ \neq 0}}^{\infty} e^{2i(\nu+\sigma)A}\times \Bigg[(I_\nu \operatorname{sgn}\eta - iJ_\nu) e^{-|\eta|\Omega_\nu} + \Big\{\beta^2\cos\gamma\cdot\operatorname{sgn}\eta + i\Big(-\beta^2\sin\gamma\cdot\operatorname{sgn}\nu + \frac{1}{|\nu|}\frac{s\lambda\beta_a^2}{2\pi\beta}\Big)\Big\}\times e^{-2|B|\operatorname{sgn}\nu(\nu+\psi)}\Bigg] \qquad \text{(B6)}$$

$$K_{vr}^{(0)}(x, y) = -\frac{i}{2s\beta_a^2}\{(1+\operatorname{sgn}\eta) P_o e^{-i\lambda\eta + iq_o y/\cos\gamma} - (P_o \operatorname{sgn}\eta + iQ_o) e^{i\phi\eta + 2i\sigma A - |\eta|\Omega_o}\} \qquad \text{(B7)}$$

$$K_{vr}^{(S)}(x, y) = \frac{i}{2s\beta_a^2}(1+\operatorname{sgn}\eta) V e^{-i\lambda\eta + iq_o y/\cos\gamma} - \frac{i}{8s\beta_a^2} e^{i\phi\eta}\big[e^{2i\sigma A - 2|B|\psi}\{(-S\cdot\operatorname{sgn}\eta + iT)\operatorname{cosech}^2(C) + 2(-V\cdot\operatorname{sgn}\eta - iW) R\coth(C) + 4U\cdot\operatorname{sgn}\eta\cdot S\coth(C)\} + e^{-2i\sigma A + 2|B|\psi}\{(S\cdot\operatorname{sgn}\eta + iT)\operatorname{cosech}^2(D) + 2(-V\cdot\operatorname{sgn}\eta + iW) R\coth(D) - 4U\cdot\operatorname{sgn}\eta\cdot S\coth(D)\}\big] \qquad \text{(B8)}$$

$$K_{vr}^{(R)}(x, y) = -\frac{i}{2s\beta_a^2}(1+\operatorname{sgn}\eta) e^{-i\lambda\eta}\sum_{\substack{\nu=-\infty\\ \neq 0}}^{\infty}\Big(P_\nu - \nu S - V - \frac{U}{\nu}\Big)\times e^{iq_\nu y/\cos\gamma} - \frac{i}{2s\beta_a^2} e^{i\phi\eta}\sum_{\substack{\nu=-\infty\\ \neq 0}}^{\infty} e^{2i(\nu+\sigma)A}\times \Bigg[(-P_\nu\cdot\operatorname{sgn}\eta - iQ_\nu) e^{-|\eta|\Omega_\nu} - \Big\{\operatorname{sgn}\eta\cdot\Big(-\nu S - V + \frac{U}{\nu}\Big) + i\operatorname{sgn}\nu\cdot(\nu T + W)\Big\} e^{-2|B|\operatorname{sgn}\nu\cdot(\nu+\psi)}\Bigg] \qquad \text{(B9)}$$

Here

$\psi = \sigma - s\lambda M_a^2\tan\gamma/(2\pi\beta^2\cos\gamma)$, $C = |B| - iA$,

$D = |B| + iA$, $R\coth(x) = \coth x - 1$,

$S\coth(x) = x - \log(\sinh x) - \log 2$,

$I_\nu = (-q_\nu^2\beta^2\cos\gamma - 2q_\nu\beta^2\lambda\sin\gamma\cdot\cos\gamma + \lambda^2 M_a^2\tan\gamma\cdot\sin\gamma)/\{(q_\nu + \lambda\sin\gamma)^2 + \lambda^2\cos^2\gamma\}$,

$J_\nu = [-q_\nu^3\beta^2\sin\gamma\cdot\cos\gamma + q_\nu^2\lambda\{\tan\gamma\cdot\sin\gamma(3M_a^2 - 2\cos^2\gamma) + \beta_a^2\cos\gamma\} + 3q_\nu M_a^2\lambda^2\tan\gamma\sin^2\gamma + M_a^2\lambda^3\tan\gamma\cdot\sin\gamma]/[\beta_a^2\Omega_\nu\{(q_\nu + \lambda\sin\gamma)^2 + \lambda^2\cos^2\gamma\}]$,

$P_\nu = \tan\gamma\cdot\Omega_\nu J_\nu - (q_\nu\beta^2\cos\gamma - M_a^2\lambda\tan\gamma) I_\nu/\beta_a^2$,

$Q_\nu = \tan\gamma\cdot\Omega_\nu J_\nu + (q_\nu\beta^2\cos\gamma - M_a^2\lambda\tan\gamma) J_\nu/\beta_a^2$,

$U = s\lambda^2\beta_a^2/(2\pi)$,

$V = 2\pi\{\beta^2(\beta^2\cos^2\gamma - \sin^2\gamma)(\sigma + s\lambda\sin\gamma/2\pi) + s\lambda\sin^3\gamma/\pi\}/(\beta_a^2 s)$,

$W = 2\pi\beta\cos\gamma\{-2\beta^2\sin\gamma(\sigma + s\lambda\sin\gamma/2\pi) + (\beta^2\cos^2\gamma + 3\sin^2\gamma)s\lambda/2\pi\}/(\beta_a^2 s)$,

$S = 2\pi\beta^2(\beta^2\cos^2\gamma - \sin^2\gamma)/(\beta_a^2 s)$,

$T = 4\pi\beta^3\sin\gamma\cdot\cos\gamma/(\beta_a^2 s)$.

Appendix C Justification of the Perturbation Treatment

Let the undisturbed velocity, density and pressure be denoted by $\mathbf{U}_o$, ρ_o, and p_o respectively. Further we denote the time mean parts of the disturbance velocity, density, and pressure by $\bar{\mathbf{u}}$, $\bar{\rho}$, and $\bar{p}$, respectively, and the unsteady parts by $\tilde{\mathbf{u}}$, $\tilde{\rho}$, and $\tilde{p}$, respectively. We regard the unsteady quantities as one order smaller than the steady quantities. Then the zeroth-order Euler's equation of motion is

$$\rho_o(\mathbf{U}_o\cdot\nabla)\bar{\mathbf{u}} + \nabla\bar{p} - \bar{\mathbf{F}} = -\rho_o(\bar{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} - \bar{\rho}([\mathbf{U}_o + \bar{\mathbf{u}}]\cdot\nabla)\bar{\mathbf{u}} \qquad \text{(C1)}$$

and the first-order equation is

$$\rho_o\{\partial/\partial t + (\mathbf{U}_o\cdot\nabla)\}\tilde{\mathbf{u}} + \nabla\tilde{p} - \tilde{\mathbf{F}} = -\rho_o\{(\bar{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}} + (\tilde{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}}\} - \bar{\rho}\{\partial/\partial t + ([\mathbf{U}_o + \bar{\mathbf{u}}]\cdot\nabla)\}\tilde{\mathbf{u}} - \bar{\rho}(\tilde{\mathbf{u}}\cdot\nabla)\bar{\mathbf{u}} - \tilde{\rho}([\mathbf{U}_o + \bar{\mathbf{u}}]\cdot\nabla)\bar{\mathbf{u}} \qquad \text{(C2)}$$

where t denotes the time, and $\bar{\mathbf{F}}$ and $\tilde{\mathbf{F}}$ denote the steady and unsteady body forces representing the blade force.

In the present problem the magnitude of each property may be estimated as follows:

$$\bar{\mathbf{u}} \sim U_o\bar{\alpha}_{-\infty}, \quad \bar{p} \sim \rho_o U_o^2\bar{\alpha}_{-\infty}, \quad \bar{\rho} \sim \bar{p}/a_o^2,$$
$$\tilde{\mathbf{u}} \sim a^*\omega, \quad \tilde{p} \sim \rho_o U_o a^*\omega, \quad \tilde{\rho} \sim \tilde{p}/a_o^2$$

where $\bar{\alpha}_{-\infty}$, a^*, ω, and a_o are the mean angle of attack, the amplitude of the blade displacement, the angular frequency of the blade vibration and the velocity of sound of the undisturbed fluid, respectively. According to the assumption 4 the terms on the right-hand side of Eq. (C1) should be sufficiently small compared to the largest term on the left-hand side. This requirement can be fulfilled if $\bar{\alpha}_{-\infty}$ is sufficiently small compared to unity. Then it follows that the terms on the right-hand side of Eq. (C2) also can be omitted, since the ratios of those to $\rho_o(\mathbf{U}_o \cdot \nabla)\tilde{\mathbf{u}}$ on the left-hand side are at most of order $\bar{\alpha}_{-\infty}$. Similar estimation can be made for the continuity and energy equations, too. Consequently we can reach the linearized steady and unsteady equations of acoustic type adopted in the text.

Finally, it remains to consider the order of magnitude of the unsteady body force $\tilde{\mathbf{F}}$ which is composed of the effect of the unsteady lift force and the effect of the blade displacement with finite lift force. We denote the former by $\tilde{\mathbf{F}}_d$ and the latter by $\tilde{\mathbf{F}}_q$. Then we can estimate them as follows

$$\tilde{\mathbf{F}}_d \sim \rho_o U_o a^* \omega / c, \quad \tilde{\mathbf{F}}_q \sim \rho_o U_o^2 \bar{\alpha}_{-\infty} a^*/c^2$$

Therefore $\tilde{\mathbf{F}}_q/\tilde{\mathbf{F}}_d \sim \bar{\alpha}_{-\infty}/(\omega c/U_o)$. Here c denotes the blade chord. We see that $\tilde{\mathbf{F}}_q$ is of the same order as $\tilde{\mathbf{F}}_d$ if the reduced frequency $\omega c/U_o$ is of the same order as $\bar{\alpha}_{-\infty}$.

To conclude, the perturbation adopted in the text is rational when $\bar{\alpha}_{-\infty}$ is small and at the same time the reduced frequency is as small as $\bar{\alpha}_{-\infty}$.

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Screen Nozzles for Gasdynamic Lasers

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Fine-hole orifice plates (screen nozzles) are considered as a replacement for two-dimensional grid nozzles in gasdynamic lasers. Analyses are made of the downstream fully mixed state, the flow perturbation decay to that state, and the optical gain distribution for typical N_2-CO_2 laser conditions. Experiments were conducted with a screen nozzle mounted on a shock tube. Density nonuniformities were found to be below 1% at 100 orifice spacings downstream (30 cm), with gains of 0.6–0.8%/cm. These results agree with predictions and lead to the conclusion that screen nozzles offer extreme simplicity with only small performance penalty.

I. Introduction

A HIGH degree of flow uniformity is required in the cavity of a modern gasdynamic laser (GDL) if significant distortion of the extracted beam is to be avoided. This has led to considerable study of the aerodynamic details of the flow. A review of some of this work[1] points out that GDL grid nozzles must be designed by the method-of-characteristics and must have carefully selected entrance profiles,[2] contoured and flat-wall boundary-layer corrections,[3] and nozzle trailing edge truncation compromising conflicting requirements of strength and flow uniformity.[4] Further, high operating pressure levels lead directly to small nozzle scales (0.01–0.02 cm throat height), and to machining and assembly tolerances that are difficult and costly to meet.

The present paper considers the feasibility of replacing the standard array of these nozzles with an orifice plate containing simple drilled holes of specified geometry and size. From strictly dimensional reasoning, the disturbance level due to the nonideal nature of the orifice flow would be expected to decay with a downstream distance measured in terms of the characteristic orifice spacing. Thus, with a sufficiently fine scale "screen nozzle," it should be possible to achieve adequate flow uniformity at a distance downstream that is still acceptable for the nonequilibrium aspects of a GDL.

Earlier studies of screen nozzles for supersonic wind tunnels[5] identified nonideal features of the downstream flow. These were waves arising from impingement of the jets from adjacent nozzles, the wake structure, and waves originating at the duct walls. The present study considers disturbances in the body of the flow in more detail, with emphasis on the N_2-CO_2 GDL

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Index categories: Lasers; Jets, Wakes, and Viscid-Inviscid Flow Interactions; Nozzle and Channel Flows.

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